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Modified Sparse Time Domain Technique for Rotor Stability Testing

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Because of nonlinear and complex interactions involved with the dynamic and aeroelastic stability of a rotor system, detailed and accurate knowledge of the stability characteristics are essential to prove design safety and to validate theoretical analysis. The task of determining these characteristics for a rotor from tests becomes complicated because of the presence of substantial amplitudes of the undamped harmonics, the possibility of close modes, and the complexity associated with the excitation of the modes in the rotating environment. Time domain modal parameter estimation methods are very useful for damping estimation when there are close modes, but they are very sensitive to noise. Subspace methods substantially improve the time domain estimates for noisy data, but they require higher computation time. A simple method is developed that retains the low variance estimation property of the subspace methods, but which is comparable in computation cost to methods that do not use the subspace approach. Its performance is evaluated for multi-output and single-output implementations and compared to the standard sparse time domain method. It is found that the modified method is more accurate in terms of the bias and standard deviation of the damping estimates, and it is faster when the number of modes is much less than the order of the data matrix.

Introduction

DYNAMIC instability of a helicopter rotor system may lead to a catastrophe. This factor has led to a significant amount of theoretical and experimental studies to better understand and control the aeroelastic behavior of such rotors. Whether it is a scaled-model test in the wind tunnel or the flight test of a full-scale vehicle, testing time is a critical factor because of the high cost and safety hazard associated with rotor stability measurement. In a dynamic stability test of a rotor, the damping and frequency of different vibration modes are determined. The task of determining these characteristics for a rotor becomes complicated because of the presence of substantial amplitudes of undamped harmonics, the possibility of close modes, and the complexity associated with the excitation of modes in the rotating environment. Because of nonlinear and complex interactions involved with the phenomenon of the stability of a rotor system, measurement of detailed and accurate stability characteristics are important to prove design safety and to validate theoretical analyses. In addition, on-line data reduction is necessary to reduce test time and to give insight into incipient damaging instability. This paper will address these points by formulating a modified sparse time domain (STD) technique to estimate stability characteristics in a rotating environment using multiple transient response of a rotor.

Of the various methods adapted to rotor stability testing, the moving-block analysis technique[1,2] is the most widely used. It is a single-mode frequency-domain modal parameter estimation technique for analyzing transient response data. Extensive evaluation and improvements of this method were described in Ref. 3, including time domain windowing of the transient responses. It is a simple method to implement, and the damping estimates of individual modes making up a response can be determined separately using rapid Fourier analysis techniques. Johnson[4] analyzed the performance of the moving-block technique and showed that it displays good statistical behavior for a single-degree-of-freedom system. In general, the moving-block method performs well for data involving well-separated modes. It does not perform well for data with close modes and high damping and with large undamped rotor harmonics.

A method developed by Grant[5] identifies the periodic damping coefficient of the flap response of an articulated rotor by estimating the variance of a response produced by turbulent aerodynamic excitation. This measured variance is then used to estimate the damping coefficient. This method was, however, developed for single degree-of-freedom flap response, and it is not directly applicable to coupled and multimode responses obtained in typical rotor stability tests. Another method that has been used for damping estimation is the random decrement method developed initially by Cole[6] for fixed wing testing and applied in Ref. 1 to rotor stability testing. Ibrahim[7] extended this method to a general multi-output setting. This method makes use of random excitation of the rotor from wind turbulence, and it is based on averaging segments of the response that have the same initial conditions. Recently, Vandiver et al.[8] have shown that this method is similar to the autocorrelation technique in time series analysis. However, it requires considerable test time. Johnson[9] and Gupta[10] used the instrumental variable technique in the frequency domain to estimate the stability characteristics of a rotor system from the transfer function. This method attempts to reduce the bias of ordinary least squares estimates by using a separate set of data called instrumental variables.[11] The instrumental variables should be strongly correlated with the noise-free part of the data and uncorrelated with the noise. In general, the selection of instrumental variables is not a well-defined procedure. Several studies have been conducted by various investigators on different choices of instrumental variables.[11]

Molussis[12] applied a recursive maximum likelihood approach to the identification of a single degree-of-freedom system under random excitation. The use of maximum-likelihood methods for multidegree-of-freedom responses is computationally prohibitive and also may converge to local minima.[13]

Ibrahim[14] formulated a time domain technique for estimation of modal parameters from multi-output transient data. This technique, called the Ibrahim time domain (ITD) method, consists of obtaining a system matrix of responses separated by a time interval and then solving for its eigenvalues. The per-

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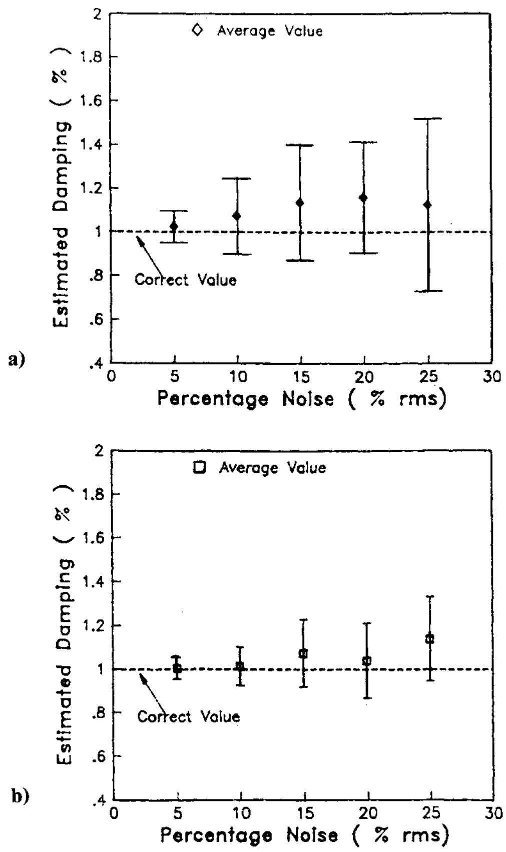


Fig. 1 Effect of noise level on damping estimates, frequency = 9.0 Hz, second mode (1/rev) at 15.0 Hz: a) STD method; b) modified STD method.

formance of the ITD technique has been extensively studied through application to a variety of structures. A closely related approach is the STD method,[15] which may also be viewed as a generalized Prony method. The STD method is a multi-output technique that incorporates the idea that all of the responses have poles corresponding to roots of the same characteristic equation. The coefficients of this characteristic equation are first determined, and then the roots are used to obtain the natural frequencies and damping values. This method uses a least squares solution and needs more modes in the estimation procedure than the number of structural modes. This means that the noise is modeled as a sum of additional modes. This can be viewed as a modification to the Prony method, and it improves the estimation of modal parameters but introduces spurious computational modes. Such extensions are, however, necessary when the data are corrupted with noise. Most time domain methods attempt to solve equations similar in structure to the ITD or STD equations. Because of noise data, different methods of solution give different statistical properties to the estimates. Some of these methods are reviewed below.

It is well known that ordinary least squares solutions are optimum when the (equation) error is Gaussian and uncorrelated, and then these become the maximum likelihood estimates. When the error is correlated, the generalized least squares (GLS) solution will give the optimum estimates, provided the correct covariance of the error is available. Iterative GLS methods have been studied in Refs. 16–19 for estimation of modal parameters in the time domain. These methods attempt to whiten the equation error. However, the iterations involve the inversion of matrices, and the computation time is increased significantly.

Recently, in the fields of modal testing and signal processing much attention has been focused on the use of subspace techniques for time domain estimation of modal parameters. These methods decompose an overdetermined, high-order response or correlation matrix into signal and noise subspaces. Either of these subspaces is then used to obtain improved estimates of the modal parameters.

An early subspace method, the Pisarenko harmonic decomposition method,[20] uses a one-dimensional null space of an autocorrelation matrix. This method is also similar to the more recent total least squares (TLS) technique.[21,22] The TLS method is a general method for solving linear equations where both the right-hand side and the left-hand side are contaminated by noise. Kumaresan[23] derived an improved Pisarenko (IP) method that produces improved parameter estimates from a noisy data matrix. This technique, as well as TLS, uses the redundancy of the noise subspace associated with an estimated covariance matrix of the data.

A notable technique for modal testing is the eigenrealization method,[24] which uses system realization concepts and a singular value decomposition (SVD) solution. Reference 24 shows that under certain conditions this method represents an SVD-based solution of the ITD equations. It is, therefore, slightly more involved computationally.

Another subspace method introduced by Kumaresan[25] uses the SVD of the response matrix. This method estimates the frequencies and damping values with significantly reduced variances. This is essentially an SVD solution of the STD equations for single-output data. In general, such time domain methods are capable of resolving very close modes. This method was used for damping estimation, from rotor stability data in Ref. 3. It was concluded that the subspace methods are necessary for damping estimation from rotor stability data with time domain methods. However, this increases the computation time, which is a critical factor in stability testing. This study is concerned with the development of a multi-output subspace method that retains the low variance property but requires much less computation time.

Analysis

To describe the modifications made to the STD method, a concise presentation of both the ITD and the STD methods follows.

Sparse Time Domain and Ibrahim Time Domain Methods

The STD technique is a multiple output, multidegree-of-freedom technique for estimating modal parameters from transient response data.[15]

The free response of a viscously damped lumped parameter system may be written as

$$x(t) = \sum_{j=1}^{2p} \psi_j e^{\lambda_j t} \tag{1}$$

where $x(t)$ is an $M \times 1$ vector of responses. An element of $x(t)$ is a response at a particular location at time t, p is the number of modes in the response, ψ_j is the jth mode shape (including the modal contribution), and λ_j is the jth eigenvalue:

$$\lambda_j = -\zeta_j \omega_j + i\omega_j \sqrt{1-\zeta_j^2} \tag{2}$$

where ζ_j is the damping ratio of jth mode and ω_j its natural frequency. Equation (1) may be written at different time instants as

$$\left[x(t_1), x(t_2), \ldots, x(t_L)\right] = \left[\psi_1, \psi_2, \ldots, \psi_l\right] \Lambda \tag{3}$$

where $\Lambda_{ij} = e^{\lambda_i t_j}$. These may be simply written as

$$\Phi = \Psi \Lambda \tag{4}$$

where Ψ is an $M \times L$ matrix and Λ an $L \times L$ matrix and has a Vandermode structure (i.e., $\Lambda_{ij} = e^{\lambda_i j \Delta t} = \Lambda_{i1}^j$).

If the response matrix is shifted by Δt, then Eq. (4) becomes

$$\Phi_s = \Psi \Lambda_s \tag{5}$$

where

$$\Lambda_s = \alpha \Lambda \tag{6}$$

Here α is diagonal and $\alpha_{ii} = e^{\lambda_i \Delta t}$, so Eq. (5) may be written as

$$\Phi_s = \Psi \alpha \Lambda \tag{7}$$

Two different approaches may be used to solve for Λ and Ψ from Eqs. (4) and (7).

In the first approach, one seeks a matrix B such that

$$B\Psi = \Psi\alpha \tag{8}$$

Then

$$\Phi_s = B\Phi \tag{9}$$

and B may be estimated using a pseudo-inverse solution:

$$B = \Phi_s \Phi^T [\Phi\Phi^T]^{-1} \tag{10}$$

This is the ITD method. To ensure invertibility of the matrix product $\Phi\Phi^T$, Φ must have linearly independent rows. The implication is that the order of the data matrix must be equal to twice the number of modes in the response. Equation (8) shows that the eigenvectors of B are the mode shapes, whereas the eigenvalues are the exponentiated poles of the system.

In the second approach, one seeks a matrix H such that

$$\Lambda H = \alpha \Lambda \tag{11}$$

This is equivalent to

$$H\Lambda^{-1} = \Lambda^{-1}\alpha \tag{12}$$

Using Eqs. (11), (5), and (4) gives

$$\Phi H = \Phi_s \tag{13}$$

H may then be estimated using a pseudo-inverse solution as

$$H = [\Phi^T\Phi]^{-1}\Phi^T\Phi_s \tag{14}$$

However, if the time shift between Φ and Φ_s is equal to the shift between adjacent columns in Φ, then the first $L-1$ columns of Φ_s are the same as the last $L-1$ columns of Φ. This means the first $L-1$ columns of H effect a column shift operation such that H is an upper Hessenberg companion matrix.[15] Thus, the last column contains the system's information. All other elements of this matrix are zero with the exception of entries on the lower diagonal, which are unity. It is then possible to solve for the last column of H alone. That is,

$$\Phi a = \phi_L \tag{15}$$

where ϕ_L is the last column of Φ_s. This is the STD method. It is also possible to write this as

$$\Phi_a b = 0 \tag{16}$$

where

$$\Phi_a = [\Phi \mid \phi_L] \text{ and } b^T = \lfloor a^T \mid -1 \rfloor$$

Φ_a is exactly like Φ but with an additional response element per location. The subscript is dropped for subsequent presentation. These equations are the same as the classical Prony method; however, this method implicitly recognizes that all of the locations satisfy the same characteristic equation. The Prony method usually refers to a single-output implementation in which each row of the data matrix is shifted in time with respect to adjacent rows. Equation (15) is often solved as a least squares solution, using Cholesky decomposition for the resulting normal equations. From Eq. (11), the eigenvalues of H are $e^{\lambda_i \Delta t}$, and the eigenvectors are the columns of $[\Lambda]^{-1}$. Alternatively, the eigenvectors of H^T are Λ^H. Using Eq. (2), the natural frequency and damping factor of each mode may be obtained from corresponding eigenvalue, and the mode shape may be determined using Eq. (4).

Modifications to the STD Method

It can be shown that the data matrix Φ and the matrix $\Phi^T\Phi$ have a rank equal to twice the number of modes present in the data if the data are free from noise. Using a higher-order matrix would, therefore, make the matrices singular. When there is noise in the data, this property is not completely true. The data matrices become ill conditioned, and the estimates obtained from repetitions of a test would exhibit substantial scatter. The scatter is not a result of numerical error but rather is due to variations in different tests. The estimates may be made robust by the use of an SVD approach[23] or by the use of principal components. The philosophy behind these approaches for improving estimates from noise-contaminated data is described next.

Consider data that are free from noise:

$$\{x(n)\} = \sum_{j=1}^{2p} \{\psi_j\} e^{\lambda_j n \Delta t} \tag{17}$$

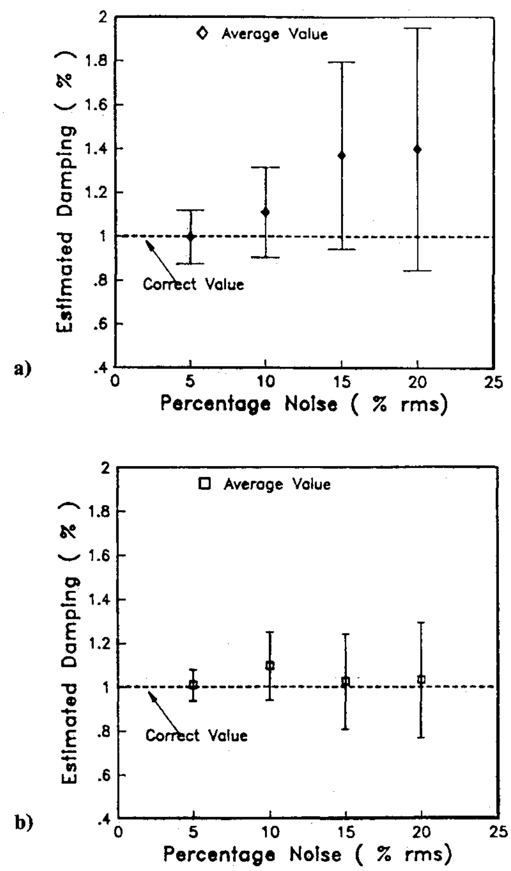


Fig. 2 Effect of noise level on damping estimates, frequency = 9.0 Hz, second mode (1/rev) at 12.0 Hz: a) STD method; b) modified STD method.

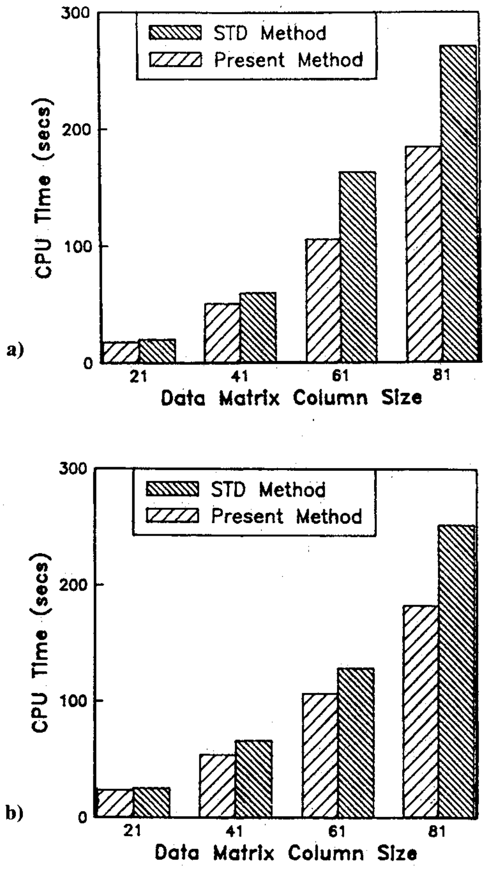


Fig. 3 Comparison of computation times of STD method and present method: a) data consisting of two modes; b) data consisting of 10 modes.

Here, $2p$ is the number of modes. The $M \times L$ data matrix Φ may be factored as

$$\Phi = \Psi\Lambda \tag{18}$$

where

$$\Phi = \begin{bmatrix} \psi_{1,1} & \cdots & \psi_{2p,1} \\ \psi_{1,2} & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \psi_{1,M} & \cdots & \psi_{2p,M} \end{bmatrix} \tag{19}$$

and

$$\Lambda = \begin{bmatrix} e^{\lambda_1 \Delta t} & \cdots & e^{\lambda_1 L \Delta t} \\ e^{\lambda_2 \Delta t} & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ e^{\lambda_{2p} \Delta t} & \cdots & e^{\lambda_{2p} L \Delta t} \end{bmatrix} \tag{20}$$

This means that

$$\Phi^T\Phi = \Lambda^H \Psi^H \Psi \Lambda \tag{21}$$

Following Ref. 22, if $\Psi^H\Psi\Lambda\Lambda^H$ has eigenvectors q_i and eigenvalues β_i $(i = 1, 2, \ldots, 2p)$, then

$$\Psi^H\Psi\Lambda\Lambda^H q_i = \beta_i q_i \tag{22}$$

Premultiplying by Λ^H gives

$$\Lambda^H\Psi^H\Psi\Lambda(\Lambda^H q_i) = \beta_i(\Lambda^H q_i) \tag{23}$$

The eigenvectors of $\Phi^T\Phi$ are $\Lambda^H q_i$, and the corresponding eigenvalues are β_i. This implies that an eigenvector of $\Phi^T\Phi$ is a linear combination of the $2p$ rows of Λ and that all such eigenvectors would span a space of dimension $2p$, the same as the rows of Λ. That is,

$$[v_1, v_2, \ldots, v_{2p}] = [\lambda_1^H, \lambda_2^H, \ldots, \lambda_{2p}^H]D$$

or

$$V = \Lambda^H D \tag{24}$$

where λ_i refers to the ith row of Λ as defined in Eq. (19). Also, note that for noise-free data there would only be $2p$ nonzero eigenvalues of the $\Phi^T\Phi$ matrix.

To illustrate the filtering effect of a subspace method, consider the data matrix corrupted by additive white noise of variance σ^2:

$$\Phi = \Psi\Lambda + E \tag{25}$$

where E is given by

$$\begin{bmatrix} e_1(1) & \cdots & e_1(L) \\ e_2(1) & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ e_M(1) & \cdots & e_M(L) \end{bmatrix} \tag{26}$$

If the row dimension M becomes very large, the elements of $\Phi^T\Phi/M$ become close to the autocorrelation of the data. In the probability limit

$$\Phi^T\Phi/M = \Lambda^H\Psi^H\Phi\Lambda/M + \sigma^2 I \tag{27}$$

Here it is assumed that the noise is white and is, therefore, uncorrelated with response data. As shown previously, the first term on the right-hand side of Eq. (27) has rank $2p$, and it is spanned by $2p$ orthonormal vectors. One may write, therefore,

$$\Phi^T\Phi = \sum_{i=1}^{2p} \beta_i v_i v_i^T + \sigma^2 I \tag{28}$$

where scaling by M is now incorporated in the definition of $\Phi^T\Phi$. From the orthonormal structure of the eigenvectors of $\Phi^T\Phi$, one obtains

$$I = \sum_{k=1}^{L} v_k v_k^T \tag{29}$$

and

$$\Phi^T\Phi = \sum_{i=1}^{2p} (\beta_i + \sigma^2) v_i v_i^T + \sum_{i=2p+1}^{L} \sigma^2 v_i v_i^T \tag{30}$$

This eigenvalue shows that the first $2p$ eigenvalues of $\Phi^T\Phi$ would be $\beta_i + \sigma^2$ and that the others would be σ^2. Also, the principal vectors corresponding to the first $2p$ eigenvalues are the same as those of Eq. (24). It is now possible to construct the noise-free data matrix by using only the $2p$ principal vectors and correcting the eigenvalue weighting (since σ^2 has been obtained). This matrix can be obtained from the first term in Eq. (30) and, in effect, is a filtered data matrix. The noise-free data matrix can then be used to estimate the true modes accurately. In actual practice, the amount of data available is finite and the properties described previously can only be applicable approximately. Thus, the noise-related eigenvalues will not have the same magnitude, but the variations would tend to be small. It also implies that the columns of V would only be approximately represented by $2p$ columns of Λ^H, as in Eq. (24). Also, from Eqs. (16) and (4):

$$\Psi\Lambda b = 0$$

$$\Psi^H\Psi\Lambda b = 0$$

$$\Lambda b = 0 \tag{31}$$

For this equation, the nonsingular nature of $\Psi^H\Psi$ is used. This equation shows that $e^{\lambda_i \Delta t}$ are roots of an Lth-order polynomial formed from the elements of b. Also, transposing Eq. (24) and postmultiplying the b gives

$$V^T b = D^H \Lambda b = 0 \tag{32}$$

This is the improved Pisarenko (IP) method of Ref. 23, where it was derived differently for a single-output response. It is, however, quite applicable to a multi-output response.

This method shows how the system vector a (contained in b) may be estimated using $2p$ principal vectors. The roots of the monic polynomial formed from a may also be obtained as eigenvalues of a companion matrix H or H^T.

These equations are underdetermined, and a minimum norm solution (using $b_{L+1} = -1$) can be easily found as

$$a = V_r [V_r^T V_r]^{-1} v_{L+1} \tag{33}$$

where v_{L+1} is the last column of V^T.

Calculation of the roots of an Lth-order polynomial or, equivalently, the eigenvalues of a companion matrix is too time consuming when viewed in the context of on-line stability testing of helicopter rotors. It was shown in Ref. 3 that a partial eigensolution method may be used to extract a subset of the roots from the companion matrix eigenvalue problem. The performance of partial eigensolution methods of general (not necessarily symmetric) matrices is not as fully developed as those for symmetric matrices; however, the methods are similar. The simultaneous vector iteration method consists of a premultiplication phase, in which a trial set of vectors is multiplied with the matrix, and a reorientation phase, in which an interaction matrix is calculated and its eigensolution is obtained. Convergence of the partial eigensolution terminates the procedure. Consider the eigenvalue problem of the companion matrix H^T obtained from noise-contaminated data:

$$H^T \Lambda_1^H = \Lambda_1^H \alpha \tag{34}$$

A trial set of iteration vectors Q may be expanded as

$$Q = \Lambda_1^H C = \Lambda_{1S}^H C_S + \Lambda_{1N}^H C_N \tag{35}$$

Here, Λ_{1S}^H refers to the eigenvectors of H^T corresponding to its largest eigenvalues, and Λ_{1N}^H refers to the eigenvectors associated with the smaller eigenvalues. This implies that

$$H^T Q = \Lambda_{1S}^H \alpha_{1S} C_S + \Lambda_{1N}^H \alpha_{1N} C_N = Y \tag{36}$$

The next step in the simultaneous vector iteration method involves the computation of an interaction matrix B from

$$QB = Y \tag{37}$$

The eigensolution of B is then calculated. If convergence of this eigensolution is not obtained, B is recalculated by replacing Q by Y in Eq. (36). Provided the α_{1S} are well separated from the α_{1N}, the contribution of the first term in Eq. (36) would eventually dominate, and the correct eigensolution would be obtained.

To accelerate convergence, one could begin the iteration with a set of vectors that approximately spans the same space as eigenvectors Λ_{1S}^H associated with the largest eigenvalues. This is the same as saying the elements of C_N are very small compared to those of C_S. When a backward prediction SVD method is used to obtain H, then all of the eigenvalues associated with the structural modes are outside or on the unit circle, and the noise modes are inside the unit circle.[25] This method ensures that the largest eigenvalues of H^T belong to the structural modes, and, hence, Λ_{1S}^H is associated with the structural modes. It is, therefore, desirable for fast convergence to obtain starting vectors that are linear combinations of the eigenvectors corresponding to the structural modes only. Consider the first L rows of V. These may be written as

$$V_r = \Lambda_{2S}^H D \tag{38}$$

where Λ_{2S}^H consists in an approximate sense of the first L rows of the $2p$ vectors in Λ^H that are all associated with structural modes. This implies that the principal eigenvectors of $\Phi^T\Phi$ are approximate linear combinations of the eigenvectors corresponding to the largest eigenvalues of H^T, which is the required property for good iteration vectors. In particular (if $\Lambda_{2S} = \Lambda_{1S}$), the first iteration gives

$$H^T V_r = H^T \Lambda_{1S}^H D = \Lambda_{1S}^H \alpha_{1S} D \tag{39}$$

The second stage in the first iteration involves computing an interaction matrix B as the solution of

$$V_r B = \Lambda_{1S}^H \alpha_{1S} D = Y \tag{40}$$

or equivalently

$$\Lambda_{1S}^H D B = \Lambda_{1S}^H \alpha_{1S} D \tag{41}$$

Because $\Lambda_{1S}\Lambda_{1S}^H$ is nonsingular, this is the same as

$$DB = \alpha_{1S} D \tag{42}$$

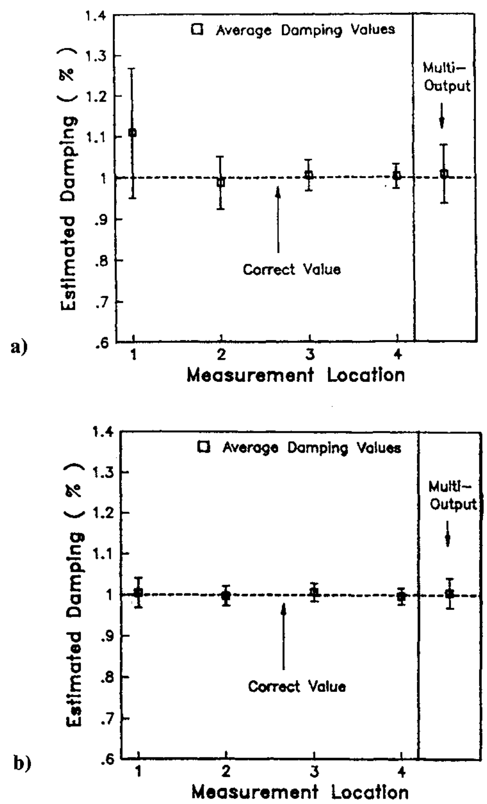
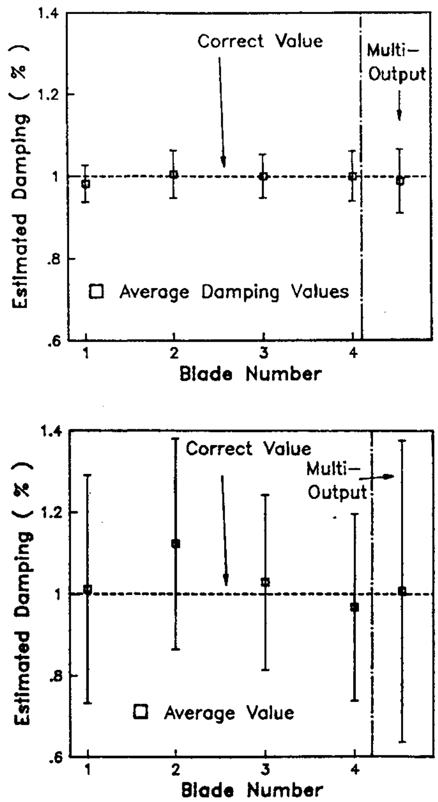


Fig. 4 Modified STD method—comparison of multi-output and single-output implementation for simulated cantilever beam responses, noise level = 25%: a) first mode ($f_1 = 9.0$ Hz); b) second mode ($f_2 = 15.0$ Hz).



a)

b)

Fig. 5 Modified STD method—comparison of multi-output and single-output damping estimates for simulated rotor transient response, low damped case ($\zeta = 0.01$), frequency of interest = 9.0 Hz, 1/rev at 15.0 Hz: a) 25% noise level; b) 80% noise level.

Equation (42) shows that the eigenvalues of B would be equal to α_{1S} when B is obtained from Eq. (40). Another method of computing only a few eigenvalues may be obtained by using the ITD method on the principal vectors. These vectors may be written as

$$V_r^T = D^H \Lambda_{2S} \tag{43}$$

If V_s^T is composed of the last L columns of V^T and one follows the same argument used in getting Eq. (7), it is easy to show that

$$V_s^T = D^H \alpha \Lambda_r \tag{44}$$

where α is a $2p \times 2p$ diagonal matrix whose elements are the system's eigenvalues. Using the procedure described for the ITD method, one may obtain a system matrix B:

$$B V_r^T = V_s^T \tag{45}$$

The eigenvalues of B are α_i, and its eigenvectors are columns of D^H. The matrix B is a $2p \times 2p$ matrix, where $2p$ is often much less than L. It is estimated using a least squares solution that

$$B = V_s^T V_r [V_r^T V_r]^{-1} \tag{46}$$

Further savings in computation time may be obtained by using the orthonormal structure of the principal vectors, as in Ref. 22:

$$\begin{aligned} V_r^T V_r &= V^T V - v_{L+1} v_{L+1}^T \\ &= I - v_{L+1} v_{L+1}^T \end{aligned} \tag{47}$$

Using the matrix inversion lemma gives:

$$[V_r^T V_r]^{-1} = I + \frac{v_{L+1} v_{L+1}^T}{1 - v_{L+1}^T v_{L+1}} \tag{48}$$

Depending on the number of modes being estimated, this method would result in significant saving of computation time since it involves only one outer product and one inner product.

Results and Discussion

To illustrate the performance of this method, a series of simulations were performed. Transient responses consisting of two modes were generated as

$$\begin{aligned} x(t) &= A_1 e^{\omega_1 \zeta_1 t} \cos\left(\omega_1 \sqrt{1-\zeta_1^2}\, t + \theta_1\right) \\ &+ A_2 e^{\omega_2 \zeta_2 t} \cos\left(\omega_2 \sqrt{1-\zeta_2^2}\, t + \theta_2\right) \end{aligned}$$

where $\omega_1 = 2\pi f_1$ rad/s and $\omega_2 = 2\pi f_2$ rad/s. For multi-output calculations the constants A_1, A_2, θ_1, and θ_2 may vary from one location to another. Zero-mean Gaussian white noise was added to each response and is expressed in terms of the ratio of the standard deviation of the noise to the rms value of the response. This rms value of the response is calculated only from the part of the transient response that is utilized in the estimation.

Comparison with the STD Method

First, the performance of this newly developed method is evaluated in comparison with the standard STD method. The data were generated using $A_1 = 10$, $A_2 = 5$, $f_1 = 9.0$ Hz, $f_2 = 15.0$ Hz, $\zeta_1 = 0.01$, $\zeta_2 = 0.0$, and $\theta_1 = 0.0$. These two modes are well separated from each other. The sampling frequency of 204.8 Hz was used for analysis. This can be visualized as a simulation of rotor stability test data, where the first mode represents a low damped lag mode and the second mode represents a one per revolution (1/rev) forced response at the rotor's rotational frequency. Damping estimation was performed using a 60×41 data matrix, consisting of responses from four blades in equal proportions. The four different blade responses were generated, including appropriate phase differences for the 1/rev, i.e., $\theta_2 = 0$, $\pi/2$, π, and $3\pi/2$. The transient response signals for all blades were assumed to be identical, an assumption that is strictly true in hover and approximately so in low speed forward flight. Different noise sequences were generated and added to each response. The magnitude of noise added to each response was assumed to be identical, and the signal-to-noise ratio was defined with respect to the first blade response ($\theta_2 = 0$). For each noise level, 50 repetitions of the estimation were carried out, with different seed values for the random number generator. The mean values of the estimated damping for the mode of interest (the damped mode) were then calculated, together with the standard deviation.

Figures 1a and 1b present estimated values of damping with varying levels of noise calculated, using the standard STD method and the newly developed modified STD method. As expected, the scatter of estimated values grows with the noise level; however, the size of the scatter for the unmodified STD method is much larger. There is considerable error in damping estimation at higher noise levels using the standard STD method, whereas the estimation error is somewhat smaller (about one-half) using the modified method.

Another case was analyzed for a smaller frequency separation between two modes ($f_1 = 9.0$ Hz, $f_2 = 12.0$ Hz), and the results are shown in Fig. 2. The error in estimated values and the scatter are considerably increased using the standard STD method, whereas these are small with the modified method.

The computation times of both methods were compared. CPU times were obtained on a Sun 360 workstation. Figure 3a shows the CPU time for both methods as the number of columns of the data matrix is increased and the row dimension

Table 1 Effect of principal vector matrix dimensions on estimates

Column size	Mean $\bar{f}_1$, Hz	Mean ζ_1, %	Std. dev. ζ_1, %	Mean $\bar{f}_2$, Hz	Mean ζ_2, %	Std. dev. ζ_2, %
40	9.0008	1.0487	0.1510	15.0023	1.0136	0.1238
38	8.9999	1.0568	0.1569	15.0017	1.0212	0.1272
36	8.9997	1.0564	0.1631	15.0016	1.0214	0.1299
34	8.9997	1.0531	0.1648	15.0018	1.0206	0.1329
32	8.9999	1.0498	0.1685	15.0020	1.0175	0.1375

held constant at 100. Here the data contain only two modes, which means that the modified method calculates the eigenvalues of the system from a 4×4 matrix. The STD method, on the other hand, determines the modes from matrices whose dimensions are equal to the column size (twice the number of modes used to model the data). For this case, the modified method becomes efficient as the order of the data matrix is increased. Figure 3b shows the comparison of computation times by the two methods when the number of modes are increased to 10. Again, it shows superior computational efficiency of the present method. However, if the number of principal vectors (twice the number of true modes) approaches the column size, the computation time of the modified method would equal that of the STD algorithm. In stability testing, it is necessary to repeat the tests for statistical reasons. This situation is ideal for the modified method since the partial eigensolution in the first part of this technique can be found more rapidly by using previously obtained estimates.

Multi-Output and Single-Output Implementations

The multi-output implementation of the modified STD method makes use of several rows of data from each location and leads to a combined set of equations. The data matrix then becomes

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_k \end{bmatrix} \tag{49}$$

For an eigensolution, one needs

$$\Phi^T\Phi = \sum_{i=1}^{i=K} \Phi_i^T \Phi_i \tag{50}$$

where i represents a particular station from a total of K stations. The matrix $\Phi^T\Phi$ for the multi-output approach is an average of those from the single-output approach. The eigenvectors of this matrix are then used to estimate the modal parameters. Two cases are considered.

The first case is the typical one encountered in modal testing, where the contributions of each mode may vary substantially from one response location to another, implying that the signal-to-noise ratio of each mode changes across locations. If the maximum size of data matrix is specified, as may be required for on-line processing, it is possible to compare the multi-output estimation results with various single-output results. Consider an example of response data, consisting of two modes from four locations of a cantilever beam located at 20, 40, 60, and 80%, respectively, of the length. The modes are assigned frequencies of 9.0 and 15.0 Hz, and a damping ratio of 0.01 was used for both modes. For multi-output implementation, a 200×41 data matrix was formed, consisting of 50 rows from each location. The estimation was carried out with data from each response location, using the same data matrix size of 200×41. Results for both modes are shown in Fig. 4. The estimates from individual locations using a single-output scheme are generally good except from location 1 (20% position), where the signal-to-noise ratio is low for the first mode. A considerable scatter in the multi-output estimation may be the result of data from location 1. However, by using the initial parts of each response, the multi-output data matrix would enhance the signal-to-noise ratio. Thus, there is a need for optimum data selection when the objective is to estimate accurately frequencies and damping values from multi-output data.

The second case, which is more applicable to rotor stability testing, is one in which the contributions of each mode are not much different from one location to the other. This would be possible if all of the blades were instrumented at the same spanwise location with comparable responses. To understand the multi-output performance of the method described previously, a set of simulations was performed. Responses consisting of an undamped response at 15.0 Hz and a damped response at 9.0 Hz with a damping ratio of 0.01 was generated for four identical blades. Figures 5a and 5b present estimated damping obtained using single-output and multi-output schemes for noise levels of 20 and 80%. For the first example, with lower noise level, damping is predicted well by both schemes. Also, the scatter in damping estimation is quite comparable for all cases. For the second example, involving a large noise level (80%), the damping estimations deteriorate with both schemes. There is large scatter of damping estimation for all cases. Although the average value of damping is estimated satisfactorily using the multi-output approach, the scatter is quite large. This example, however, is an extreme condition, where noise dominates the response.

The estimation was also repeated for a case with significantly higher damping ($\zeta_1 = 0.05$). Figure 6 shows that, for this level of damping, a multi-output approach would provide better estimation, especially at higher noise levels. Because of higher damping, the transient dies early. With a multi-output approach, one needs to use only the initial part of the response, where the signals are not dominated by noise. A data

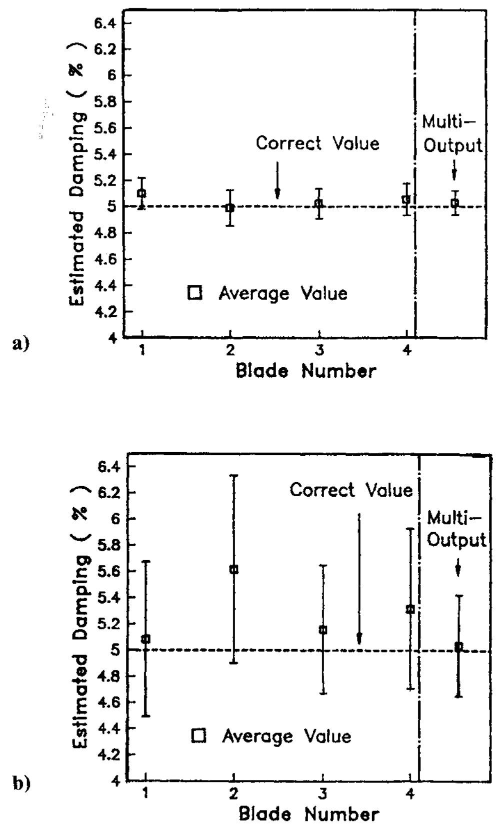


Fig. 6 Modified STD method—comparison of multi-output and single-output damping estimates, using simulated rotor transient response, high damped case ($\zeta = 0.05$), frequency of interest = 9.0 Hz, 1/rev at 15.0 Hz: a) 25% noise level; b) 80% noise level.

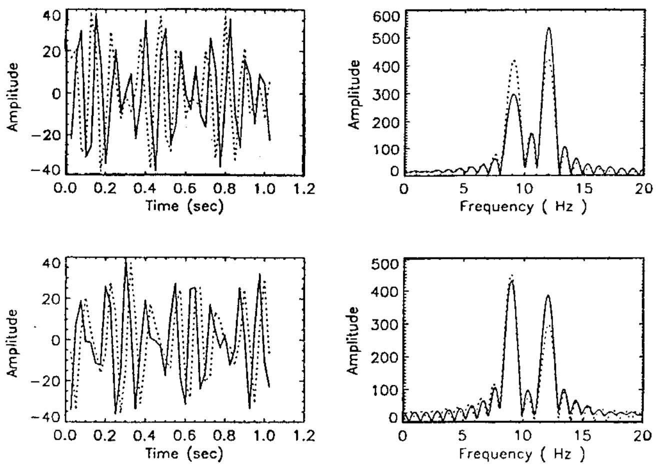


Fig. 7 Principal vectors of data consisting of damped mode at 9.0 Hz and undamped mode at 12.0 Hz, with 25% noise.

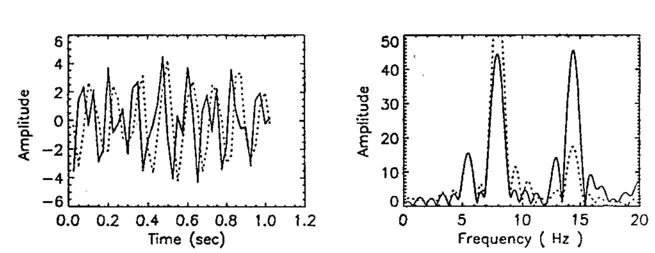


Fig. 8 Example of null space or noise eigenvector, data consisting of damped mode at 9.0 Hz and undamped mode at 12.0 Hz, with 25% noise.

matrix formed with the initial parts of the transient response from different locations would, therefore, possess a much higher signal-to-noise ratio, leading to much better damping estimates.

Characteristics of the Principal Vectors

Certain similarities are apparent between Eqs. (24) and (4), which show how the vectors v_i vary as functions of time. The time histories are useful in determining the number of significant modes in each principal vector. Figure 7 shows the amplitude and the spectrum of the principal vectors generated from a 60×41 data matrix of a transient response (as used for Fig. 2), with 25% added noise. The data were sampled at 40.0 Hz to permit a finer resolution of the vectors. It can be seen that the vectors essentially contain only the two modes of interest. Since there are only two modes, there are two pairs of principal vectors or a total of four vectors. Each pair of principal vectors is plotted using a solid line and a dotted line. These vectors may also be analyzed in the same fashion as the original data. First, an overdetermined data matrix is formed from the four principal vectors by treating them as four separate responses of a structure. In the preceding example, one would form four responses with 41 samples each. Then a singular value decomposition of this matrix is used to find its rank. While this matrix is not singular, the separation of its first four singular values from the other singular values is much more than that of the original data matrix. This means that the error in approximating the principal vectors as a sum of two modes is quite small. The error in making such an assumption for the original data is higher. This is simply a consequence of the filtering that has occurred with the subspace technique. An example of a pair of null space or noise-related principal vectors is shown in Fig. 8, which illustrates the extra modes used by the method to account for noise in the data.

The overdetermined matrix formed from the principal vectors may also be used to produce estimates of the frequencies and damping values in a way similar to the original data. An investigation into the performance of this method was undertaken. This was done by increasing the number of rows in the principal vector matrix V through the use of pseudostations. This decreases the column size (of V) while increasing the number of rows (stations). The pseudostations are time-shifted responses. The original data were generated with $f_1 = 9.0$ Hz, $f_2 = 15.0$ Hz, $\zeta_1 = \zeta_2 = 0.01$, $A_1 = A_2 = 10.0$, and $\theta_1 = \theta_2 = 0.0$. The sampling frequency was 204.8 Hz, and the noise level was 10%. The results are summarized in Table 1, with decreasing column size (or increasing row size). It can be clearly seen that no further refinement could be obtained by increasing the number of rows and decreasing the number of columns in spite of additional computation effort. These results do not follow the normal behavior for noise-contaminated data. The reason is that, although the principal vectors being used are primarily composed of two modes, they also contain several other exponentials that were used to model the noise. The contributions of these exponentials, however, are very small, and the overdetermined matrix would tend to have a rank equal to twice the number of true modes. However, the filtering effect which was demonstrated for the original data no longer applies. The noise (E) matrix would be replaced by a matrix that is a sum of small exponentials and, therefore, possesses substantial correlation, unlike the rows of E.

Conclusions

1) A multi-output modified STD method has been developed for rotor stability testing. This method is more accurate, in terms of the bias and standard deviation of the damping estimates, than the standard method and is faster when the number of modes is much less than the order of the data matrix.

2) A single pass utilization of the principal vectors in estimating the modal parameters produces highly accurate estimates. Further processing based on the assumption of noise in the principal vectors does not appear to yield any more accuracy.

3) The multi-output implementation of this method is particularly useful in rotor stability testing for cases involving high noise levels and high damping values.

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References

[1]Hammond, C. E., and Doggett, R. V., Jr., "Determination of Subcritical Damping by Moving Block/Randomdec Applications," NASA Symposium on Flutter Testing Techniques, NASA SP 415, Oct. 1975.

[2]Bousman, W. G., and Winkler, D. J., "Application of the Moving Block Analysis," *Proceedings of the AIAA/ASME/ASCE/AHS 22nd Structures, Structural Dynamics and Materials Conference* (Atlanta, GA), AIAA, New York, 1981, pp. 755–763.

[3]Tasker, F., and Chopra, I., "Assessment of Transient Testing Techniques for Rotor Stability Testing," *Journal of the American Helicopter Society,* Vol. 35, No. 1, Jan. 1990, pp. 39–50.

[4]Johnson, W., "A Discussion of Stability Measurement Techniques," NASA TM X-73,081, Nov. 1975.

[5]Grant, B. E., "A Method of Measuring Aerodynamic Damping of Helicopter Rotors in Forward Flight," *Journal of Sound and Vibration,* Vol. 3, No. 3, 1966, pp. 407–421.

[6]Cole, H. A., "On-Line Failure Detection and Damping Measurement of Aerospace Structures by Random Decrement Signatures," NASA CR-2205, March 1973.

[7]Ibrahim, S. R., "Random Decrement Technique for Modal Identification of Structures," *Journal of Spacecraft and Rockets,* Vol. 14, No. 11, 1977, pp. 696–700.

[8]Vandiver, J. K., Dunwoody, A. B., Campbell, R. B., and Cook, M. F., "A Mathematical Basis for the Random Decrement Vibration Signature Analysis Technique," *ASME Journal of Mechanical Design,* Vol. 104, April 1982, pp. 307–313.

[9]Johnson, W., and Gupta, N. K., "Transfer Function and Parameter Identification Methods for Dynamic Stability Measurement," *Proceedings of the 33rd Annual Forum of the American Helicopter Society,* American Helicopter Society, Washington, DC, May 1977, pp. 33–35.

[10]Gupta, N., and Bohn, J., "A Technique for Measuring Rotorcraft Dynamic Stability in the 40 by 80 Foot Wind Tunnel," NASA CR-151955, 1977.

[11]Young, P., *Recursive Estimation and Time Series Analysis,* Springer-Verlag, Berlin, 1984.

[12]Molussis, J. A., "Rotorcraft Blade Mode Damping Identification from Random Responses Using a Recursive Maximum Likelihood Algorithm," NASA CR 3600, 1982.

[13]Wax, M., "Detection and Estimation of Superimposed Signals," Ph.D. Dissertation, Electrical Engineering Dept., Stanford Univ., Stanford, CA, March 1985.

[14]Ibrahim, S., and Miculcik, E. C., "A Method for the Direct Identification of Vibration Parameters from the Free Response," *Shock and Vibration Bulletin,* No. 47, Pt. 4, Sept. 1977, pp. 183–198.

[15]Ibrahim, S. R., "An Upper Hessenberg Sparse Matrix Algorithm for Model Identification on Minicomputers," *Journal of Sound and Vibration,* Vol. 113, No. 1, Feb. 1987, pp. 47–57.

[16]Evans, A. G., and Fischl, R., "Optimal Least Squares Time Domain Synthesis of Recursive Digital Filters," *IEEE Transactions on Audio and Electroacoustics,* Vol. AU-21, Feb. 1973, pp. 61–65.

[17]Kumaresan, R., Scharf, L. L., and Shaw, A. K., "An Algorithm for Pole-Zero Modeling and Spectral Analysis," *IEEE Transactions on Acoustics, Speech, and Signal Processing,* Vol. ASSP-34, No. 3, June 1986, pp. 637–640.

[18]Bresler, Y., and Macovski, A., "Exact Maximum Likelihood Parameter Estimation of Superimposed Exponential Signals in Noise," *IEEE Transactions on Acoustics, Speech, and Signal Processing,* Vol. 34, No. 5, Oct. 1986, pp. 1081–1089.

[19]Matausek, M. R., Stankovic, S. S., and Radivic, D. V., "Iterative Inverse Filtering Approach to the Estimation of Noisy Sinusoids," *IEEE Transactions on Acoustics, Speech, and Signal Processing,* Vol. ASSP-31, Dec. 1983, pp. 1456–1463.

[20]Pisarenko, V. F., "The Retrieval of Harmonics from a Covariance Function," *Geophysical Journal of the Royal Astronomical Society,* Vol. 33, No. 3, 1973, pp. 347–366.

[21]Golub, G. H., and Van Loan, *Matrix Computations,* Johns Hopkins University Press, Baltimore, MD, 1983.

[22]Rahman, M. A., and Yu, K., "Total Least Squares Approach for Frequency Estimation Using Linear Prediction," *IEEE Transactions on Acoustics, Speech, and Signal Processing,* Vol. ASSP-35, No. 10, Oct. 1987, pp. 1440–1454.

[23]Kumaresan, R., and Tufts, D., "Estimating the Parameters of Exponentially Damped Sinusoids and Pole Zero Modeling in Noise," *IEEE Transactions on Acoustics, Speech, and Signal Processing,* Vol. 30, No. 6, Dec. 1982, pp. 833–840.

[24]Juang, J., and Pappa, R. S., "An Eigensystem Realization Algorithm for Modal Parameter Identification and Model Reduction," *Journal of Guidance, Control, and Dynamics,* Vol. 8, No. 5, 1985, pp. 620–627.

[25]Kumaresan, R., "Estimating the Parameters of Exponentially Damped and Undamped Sinusoidal Signals," Ph.D. Dissertation, Dept. of Electrical Engineering, Univ. of Rhode Island, Kingston, RI, Aug. 1982.